General Relativity: Tutorial 4

Professor Peter Dunsby, Room 303 Mathematics

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1. A particle of rest mass m_1 moving with velocity u_1 along the x-axis collides elastically with a stationary particle of rest mass m_2 and as a result m_1 and m_2 are deflected through angles α and β respectively. If E and E' are the total energies of the particle m_1 before and after the collision respectively, show that

$$\cos\alpha = \frac{(E + m_2c^2)E' - m_2c^2E - m_1^2c^4}{\sqrt{(E^2 - m_1^2c^4)(E'^2 - m_1^2c^4)}}$$

2. In 2D Cartesian coordinates

$$f = 2xy$$

is a scalar and

$$\mathbf{V} \to (y, xy)\ , \quad \mathbf{W} \to (1, 1)$$

are two vectors.

   (a) Compute f as a function of polar coordinates (r, θ) and find the components of $\mathbf{V}$ and $\mathbf{W}$ on the polar basis, expressing them as functions of r and θ. [**Hint: Use the transformation matrix $\Lambda^{\overline{\alpha}}{}_{\beta}$ given in the lecture notes.**]

   (b) Find the components of $\tilde{\mathbf{d}}f$ in Cartesians and obtain them in polars by (i) direct calculation in polars, and (ii) transforming from Cartesian coordinates.

   (c) Use the metric tensor in polar coordinates to find the polar components of the one-forms $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{W}}$ associated with $\mathbf{V}$ and $\mathbf{W}$. Obtain these components by transformation of the components of $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{W}}$ in Cartesians.

3. For the vector field $\mathbf{V}$ of question (1) above, compute;

   (a) $V^{\alpha}{}_{,\beta}$ in Cartesian coordinates;

   (b) the transformation $\Lambda^{\overline{\mu}}{}_{\alpha}\Lambda^{\beta}{}_{\overline{\nu}}V^{\alpha}{}_{,\beta}$ to polars;

   (c) the components $V^{\overline{\mu}}{}_{;\overline{\nu}}$ using the Christoffel symbols given in the lecture notes [Why is this the same as (b)?];

   (d) the divergence $V^{\alpha}{}_{,\alpha}$ in Cartesian coordinates;

(e) the divergence $V^{\overline{\mu}}{}_{;\overline{\mu}}$ in polars using part (c);

4. Three dimensional Cartesian coordinates (x, y, z) are related to three dimensional cylindrical polar coordinates (r, θ, z) by

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z\,.
\end{aligned}
$$

Show that the line element in cylindrical polar coordinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2\,.$$

Calculate $\Gamma^r{}_{\theta\theta}$, $\Gamma^\theta{}_{r\theta}$ - these being the only non - zero components of $\Gamma^\alpha{}_{\beta\gamma}$, and hence show that the geodesic equations in cylindrical polars are

$$
\begin{aligned}
\frac{d^2r}{d\lambda^2} &- r\left(\frac{d\theta}{d\lambda}\right)^2 = 0\,, \\
\frac{d^2\theta}{d\lambda^2} &+ \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} = 0\,, \\
\frac{d^2z}{d\lambda^2} &= 0\,.
\end{aligned}
$$

5. Using the usual coordinate transformations from Cartesian to spherical polars, calculate the metric on the surface of a sphere of unit radius. Find the inverse metric.

6. Calculate the Riemann curvature tensor of the surface of a sphere of unit radius using the result of the previous problem.